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Complex variable BEM solution for half-plane problems with straight boundary clamped

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Abstract

In this paper, a new complex variable fundamental solution which satisfies the clamped boundary conditions of half-plane problem has been derived by use of Riemann-Schwarz symmetric principle and the mathematical theory of elasticity. The correspondent complex variable boundary integral equations for elastic analysis have been given. Numerical procedure shows more efficiency and advantages of the present method over conventional boundary element method. © 1999 Elsevier Science Ltd. All rights reserved.

Keywords: Fundamental solution; Half-plane; Clamped boundary; BEM

1. Introduction

The boundary element method (BEM) for the first kind of boundary value problems (BVPs) of half-plane have been studied adequately (see e.g., Telles and Brebbia, 1981; Huang and Tang, 1986). For the second kind of BVPs, however, no literature could be found except for BEM based on Kelvin solution, which must discretize all boundaries by boundary elements. It is known that the fundamental solution is most important for BEM (Brebbia, 1980; Hromadka and Lai, 1986; Chandra and Mukhjerjee, 1997; Crouch and Starfield, 1983; Banerjee and Butterfield, 1981 among many others). Therefore, the main obstruction might be the difficulty in derivation of fundamental solutions for the above-mentioned problem. In this paper, a very concise method for deriving the complex variable fundamental solutions for second kind of BVPs of half-plane has been developed by extending the method for first kind of BVPs proposed by Huang and Tang (1986). Based on the new fundamental solution presented in this paper, the correspondent complex variable boundary integral equation method has been established. Numerical procedure and examples show the accuracy of the developed fundamental solution and the efficiency and advantages of the present complex variable BEM over conventional BEM.

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2. Basic theorems

Riemann–Schwarz symmetry principle for straight boundary may be expressed as following (Muskhelishvili, 1953, Markushevich, 1983):

Theorem 1: Let Γ be a straight line (say, $x_2 = 0$), which divides the complex plane Ω into two half-planes Ω^+ and Ω^- (Fig. 1). Suppose that $f(z)$ is an analytic function defined in one of two regions say Ω^+, and there exists boundary value $f^+(t)$, $t \in \Gamma$. Then $\bar{f}(z)$ is an analytic function in Ω^-, and there exists $\overline{f^-}(t)$ which satisfies $\overline{f^+(t)} = \overline{f^-}(t)$, $t \in \Gamma$.

The following theorem may be derived by Theorem 1 (Huang et al., 1986):

Theorem 2: If $f_0(z)$ and $F_0(z)$ are functions which are analytic in Ω^+ and continuous on $\Omega^+ + \Gamma$, then there exists functions $f_1(z)$ and $F_1(z)$ which are analytic on Ω^- and continuous on $\Omega^- + \Gamma$, and the functions

$$f(z) = f_0(z) + f_1(z) \tag{a}$$

$$F(z) = F_0(z) + F_1(z) \tag{b}$$

satisfy boundary condition:

$$\overline{f(t)} + F(t) = 0, \quad t \in \Gamma \tag{c}$$

and also when Γ is a straight line ($x_2 = 0$), one has

$$f_1(z) = -\overline{F_0}(z), \quad F_1(z) = -\overline{f_0}(z). \tag{d}$$

Proof: As $f_0(z)$ and $F_0(z)$ are analytic in Ω^+ and continuous on $\Omega^+ + \Gamma$, we know from theorem 1 that $\overline{f_0}(z)$ and $\overline{F_0}(z)$ are analytic in Ω^- and there exist boundary value $\overline{f_0^-}(t)$ and $\overline{F_0^-}(t)$ which satisfy $\overline{f_0^+(t)} = \overline{f_0^-(t)}$ and $\overline{F_0^+(t)} = \overline{F_0^-(t)}$. Thus, if one takes $f_1(z) = -\overline{F_0}(z)$ and $F_1(z) = -\overline{f_0}(z)$, then $f(z)$ and $F(z)$ of expression (a) and (b) satisfy boundary condition (c).

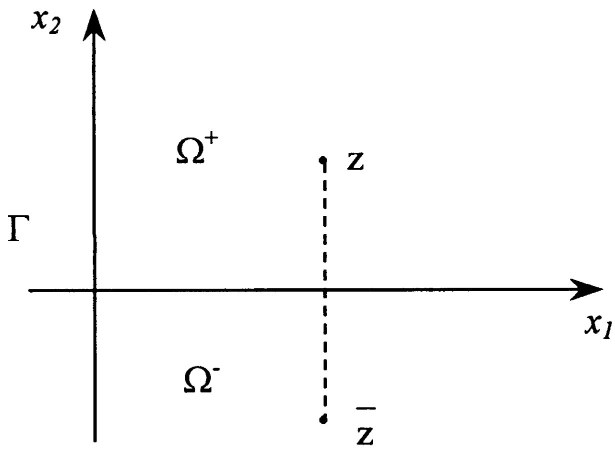

Fig. 1. Notation for half-plane.

3. The complex variable fundamental solution for half-plane problem with straight boundary clamped

The homogeneous boundary condition of the second kind of BVPs can be expressed by use of complex variable fundamental solutions as BVP1 (Muskhelishvili, 1953; Markushevich, 1983; Timoshenko and Goodier, 1970):

$$\varphi_l(z,z_0)-[\overline{z\varphi_l'(z,z_0)}+\overline{\psi_l(z,z_0)}]/(3-4v) = 0, \quad z\to\Gamma \tag{1}$$

where $\varphi_1(z, z_0)$ and $\psi_l(z, z_0)$ are complex variable fundamental solutions; $(2)' = \mathrm{d}(\,)/\mathrm{d}z$;$z$ and z_0 is field point and source point respectively (Fig. 2). Let

$$Fl(z,z_0) = -[\bar{z}\varphi_l'(z,z_0)+\psi_l(z,z_0)]/(3-4v) \tag{2}$$

then, the BVP1 can be simplified as the following BVP2:

$$\varphi_l(z,z_0)+\overline{F_l(z,z_0)} = 0, \quad z\to\Gamma \tag{3}$$

Thus, if

$$\begin{aligned}\varphi_l(z,z_0) &= \varphi_{pl}(z,z_0)+\varphi_{bl}(z,\overline{z_0})\\ \psi_l(z,z_0) &= \psi_{pl}(z,z_0)+\psi_{bl}(z,\overline{z_0})\end{aligned} \tag{4}$$

is the solution of BVP1, then

$$\begin{aligned}\varphi_l(z,z_0) &= \varphi_{pl}(z,z_0)+\varphi_{bl}(z,\overline{z_0})\\ F_l(z,z_0) &= F_{pl}(z,z_0)+F_{bl}(z,\overline{z_0})\end{aligned} \tag{5}$$

is the solution of BVP2, where

$$\begin{aligned}F_{pl}(z,z_0) &= -[\bar{z}\varphi_{pl}'(z,z_0)+\psi_{pl}(z,z_0)]/(3-4v)\\ F_{bl}(z,\bar{z}_0) &= -[\bar{z}\varphi_{bl}'(z,\overline{z_0})+\psi_{bl}(z,\overline{z_0})]/(3-4v)\end{aligned} \tag{6}$$

where $\varphi_{pl}(z, z_0)$ and $\psi_{pl}(z, z_0)$ are complex variable fundamental solutions for infinite plane at point

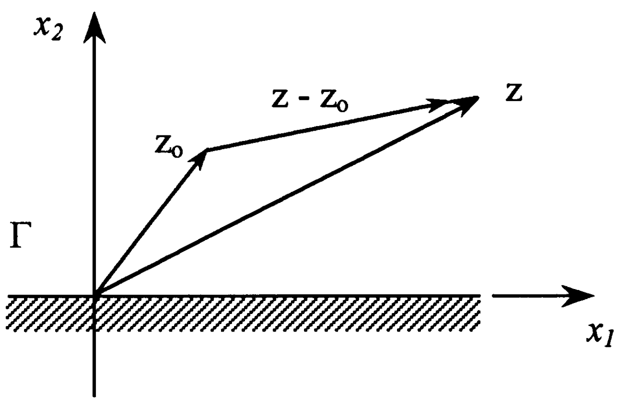

Fig. 2. Half-plane with straight boundary clamped.

z due to a unit force at point z_0; $\varphi_{pl}(z, \overline{z_0})$ and $\psi_{pl}(z, \overline{z_0})$ are mapping functions about boundary Γ to satisfy specified boundary conditions. Then by Theorem 2 we have

$$\begin{aligned} \varphi_{bl}(z,\overline{z_0}) &= -\overline{F_{pl}}(z,\overline{z_0}) \\ F_{bl}(z,\overline{z_0}) &= -\overline{\varphi_{pl}}(z,\overline{z_0}) \end{aligned} \tag{7}$$

Substituting formula (6) into (7), one obtains

$$\begin{aligned} \varphi_{bl}(z,\overline{z_0}) &= [z\overline{\phi'_{pl}}(z,\overline{z_0})+\overline{\psi_{pl}}(z,\overline{z_0})]/(3-4v) \\ \psi_{bl}(z,\overline{z_0}) &= (3-4v)\overline{\varphi_{pl}}(z,\overline{z_0})-\bar{z}\varphi'_{bl}(z,\overline{z_0}) \end{aligned} \tag{8}$$

It can be found that for given $\varphi_{pl}(z, z_0)$ and $\psi_{pl}(z, z_0)$, the mapping function $\varphi_{bl}(z, \overline{z_0})$ and $\psi_{pl}(z, \overline{z_0})$ can be obtained by formula (8). Thus, the complex variable fundamental solution φ_l and ψ_l are then given by expression (4).

Considering the symmetry of displacements, the complex variable fundamental solution $\varphi_{pl}(z, z_0)$ and $\psi_{pl}(z, z_0)$ for infinite plane problem can be derived by the solution of reference-8 (Muskhelishvili, 1953) as follows:

$$\begin{aligned} \varphi_{pl}(z,z_0) &= A_l\ln(z-z_0) \\ \psi_{pl}(z,z_0) &= B_l\ln(z-z_0)-\frac{A_l\overline{z_0}}{z-z_0}+\overline{A_l} \end{aligned} \tag{9}$$

where $A_1 = [i^{l+1}/(8\pi(1-v))]$; $B_l = i^{2l}(3-4v)A_l$; $i = \sqrt{-1}$; v is Poisson's ratio. The well known Kelvin solution can be derived from solution (9).

Substituting solution (9) into (8), paying attention to the symmetry condition of displacements and the stresses vanishing at the pole point at infinity, and remaining the principle part of functions at pole point at infinity, the mapping function can be derived as follows:

$$\begin{aligned} \varphi_{bl}(z,\overline{z_0}) &= -A_l\ln(z-\overline{z_0})-\frac{\bar{A}_l(z-z_0)}{(3-4v)(z-\overline{z_0})}-A_l \\ \psi_{bl}(z,\overline{z_0}) &= -B_l\ln(z-\overline{z_0})+\frac{A_l\bar{z}}{z-\overline{z_0}}-\frac{\overline{A_l}}{3-4v}\frac{z(z_0-\overline{z_0})}{(z-\overline{z_0})^2}-2B_l \end{aligned} \tag{10}$$

The complex variable fundamental solution for half-plane problem with straight boundary clamped is given by substituting solution (9) and (10) into formula (4).

4. Complex variable boundary integral equations for elasticity

The basic integral representation for elastic analysis can be written as (Huang and Tang, 1986):

$$C_{ij}(z_0)u_i(z_0)+\int_\Gamma P_{ij}(z,z_0)u_j(z)\,\mathrm{d}\Gamma(z) = \int_\Gamma U_{ij}(z,z_0)p_j(z)\mathrm{d}\Gamma(z) + \int_\Omega U_{ik}(z',z_0)b(z')\mathrm{d}\Omega(z') \tag{11}$$

where the coefficient tensor $C_{ij}(z_0)$ can be obtained by assuming unit rigid body displacement in all directions as:

$$C_{ij}(z_0) = -\int_\Gamma P_{ij}(z,z_0)\,\mathrm{d}\Gamma(z) \tag{12}$$

$U_{ij}(z, z_0)$ and $P_{ij}(z, z_0)$ are kernel functions of the boundary integral equation, i.e., the displacement and traction, respectively at field point z in direction x_j due to a unit concentrated force at source point z_0 in direction x_i. These kernel functions will be expressed by complex variable fundamental solution $\varphi_l(z, z_0)$ and $\psi_l(z, z_0)$.

Once the boundary values $u_i(z)$ and $p_i(z)$ are found by using expression (11) over all boundary elements, internal results of displacements and stresses can then be calculated immediately by the following integral equation.

$$u_i(z_0) = \int_\Gamma U_{ij}(z,z_0)p_j(z)\,\mathrm{d}\Gamma(z) - \int_\Gamma P_{ij}(z,z_0)u_j(z)\,\mathrm{d}\Gamma(z) + \int_\Omega U_{ik}(z',z_0)b_k(z')\,\mathrm{d}\Omega(z') \tag{13}$$

Considering the strain–displacement relation for linear elasticity, the strain tensor components can be easily obtained as:

$$\varepsilon_{ij}(z_0) = \int_\Gamma B_{ij}(z,z_0)p_k(z)\mathrm{d}\Gamma(z) - \int_\Gamma T_{kij}(z,z_0)u_k(z)\mathrm{d}\Gamma(z) + \int_\Omega B_{kij}(z',z_0)b_k(z')\mathrm{d}\Omega(z') \tag{14}$$

where the third order tensor components $B_{kij}(z, z_0)$ and $T_{kij}(z, z_0)$ are given by

$$\begin{aligned} B_{kij}(z,z_0) &= \tfrac{1}{2}[U_{ik,j}(z,z_0) + U_{jk,i}(z,z_0)] \\ T_{kij}(z,z_0) &= \tfrac{1}{2}[P_{ik,j}(z,z_0) + P_{jk,i}(z,z_0)] \end{aligned} \tag{15}$$

where the ‘,’ indicates partial derivative in the form

$$(\,)_{,i} = \frac{\partial(\,)}{\partial z_0}\frac{\mathrm{d}z_0}{\mathrm{d}x_i} \tag{16}$$

The stress tensor components can be obtained in the same way as

$$\sigma_{ij}(z_0) = \int_\Gamma D_{kij}(z,z_0)p_k(z)\mathrm{d}\Gamma(z) - \int_\Gamma S_{kij}(z,z_0)u_k(z)\mathrm{d}\Gamma(z) + \int_\Omega D_{kij}(z',z_0)b_k(z')\mathrm{d}\Omega(z') \tag{17}$$

where the third order tensor components $D_{kij}(z, z_0)$ and $S_{kij}(z, z_0)$ can be expressed in terms of kernel functions $U_{ij}(z, z_0)$ and $P_{ij}(z, z_0)$ as follows:

$$\begin{aligned} D_{kij}(z,z_0) &= \lambda\delta_{ij}U_{lk,l}(z,z_0) + \mu[U_{ik,j}(z,z_0) + U_{jk,i}(z,z_0)] \\ S_{kij}(z,z_0) &= \lambda\delta_{ij}P_{lk,l}(z,z_0) + \mu[P_{ik,j}(z,z_0) + P_{jk,i}(z,z_0)] \end{aligned} \tag{18}$$

where λ and μ are Lamé’s constants.

5. Kernel functions expressed by complex variable fundamental solutions

According to the mathematical theory of elasticity (Mukhelishvili, 1953), the displacement kernel function can be expressed in terms of complex variables as

$$U_{ij}(z,z_0) = \frac{1}{2\mu}\{(3-4v)[\delta_{j1}\,\mathrm{Re}(\overline{\varphi_i(z,z_0)}) - \delta_{j2}\,\mathrm{Im}(\overline{\varphi_i(zm\,z_0)})] \\ -\delta_{j1}\,\mathrm{Re}(\bar{z}\varphi_i'(z,z_0)+\psi_i(z,z_0)) - [\delta_{j2}\,\mathrm{Im}(\bar{z}\varphi_i'(z,z_0)+\psi_i(z,z_0))]\} \quad (19)$$

The stress σ_{ij} at point z due to a unit force at z_0 in direction x_l is:

$$\sigma_{lij}(z,z_0) = \delta_{ij}\,\mathrm{Re}\,[2\varphi_l'(z,z_0)+(-1)^i(\bar{z}\varphi_l''(z,z_0)+\psi_l'(z,z_0))] \\ +(\delta_{il}\delta_{j2}+\delta_{jl}\delta_{i2})\,\mathrm{Im}\,[\bar{z}\varphi_l''(z,z_0)+\psi_l'(z,z_0)] \quad (20)$$

and then, the traction $P_{1i}(z,z_0)$ can be obtained as

$$P_{li}(z,z_0) = n_k\sigma_{lik}(z,z_0) \\ = n_j\delta_{ij}\,\mathrm{Re}\,[2\varphi_l'(z,z_0)+(-1)^i(\bar{z}\varphi_l''(z,z_0)+\psi_l'(z,z_0))] \\ +(n_l\delta_{i2}+n_2\delta_{j1})\,\mathrm{Im}\,[\bar{z}\varphi_l''(z,z_0)+\psi_l'(z,z_0)] \quad (21)$$

where n_k are the direction cosines of the outward normal n with respect to x_k-axis.

The strain kernel function $B_{ijk}(z,\, z_0)$ can be expressed in the same way, as

$$B_{ijk}(z,z_0) = \frac{1}{4\mu}\{(3-4v)[\delta_{k1}\,\mathrm{Re}(\overline{\varphi_{i,j}(z,z_0)}+(\overline{\varphi_{j,i}(z,z_0)})) \\ -\delta_{k2}\,\mathrm{Im}(\overline{\varphi_{i,j}(z,z_0)}+(\overline{\varphi_{j,i}(z,z_0)}))] \\ -[\delta_{k1}\,\mathrm{Re}(\bar{z}(\varphi_{i,j}'(z,z_0)+(\varphi_{j,i}'(z,z_0))+\psi_{i,j}'(z,z_0)+\psi_{j,i}'(z,z_0)) \\ -[\delta_{k2}\,\mathrm{Im}(\bar{z}(\varphi_{i,j}'(z,z_0)+(\varphi_{j,i}'(z,z_0))+\psi_{i,j}'(z,z_0)+\psi_{j,i}'(z,z_0))]\} \quad (22)$$

The above equations are the basic expressions of the BEM using complex variables for elastic analysis. A general-purpose computer program may be written based on the above equations. The program may be applied to different kinds of plane problems as long as the subroutines for calculating the relevant complex fundamental solutions and their derivatives are available. Thus, the computer program may contain different complex variable fundamental solutions for different problems. The above formulation is simple and straightforward as well as all fundamental solutions are expressed by only two variables φ and ψ. This characteristic facilitates easy program writing and debugging.

6. Numerical examples and discussion

A clamped infinite straight boundary half-plane with a circular opening near the boundary is shown in Fig. 3. The internal pressure is $q = 9.8$ MPa; $E = 20.374$ GPa; $v = 0.1$. The boundary element discretization by conventional BEM is shown in Fig. 4. In the present method, as the

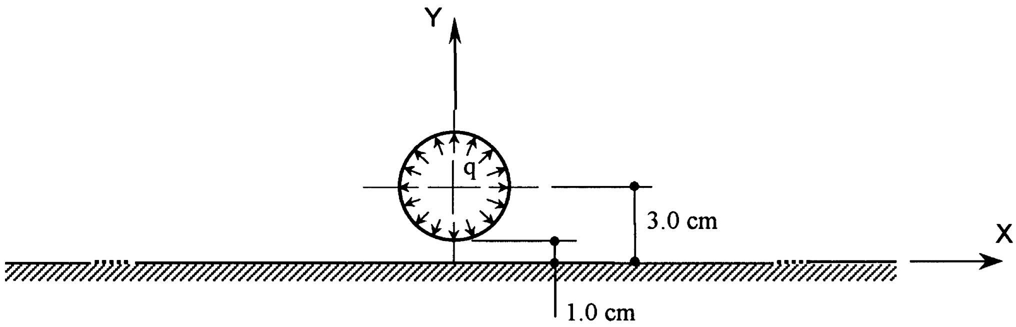

Fig. 3. Second BVP of half-plane with circular hole under internal pressure.

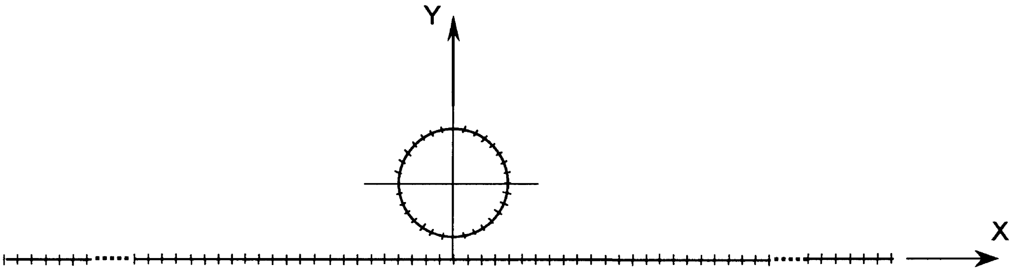

Fig. 4. Boundary element discretization by conventional BEM based on Kelvin solution.

clamped boundary conditions are satisfied by complex variable solutions, only the edge of the opening need be discretized (Fig. 5). A total of 24 constant boundary elements are employed along the opening. The stress components along the clamped boundary elements are employed along the opening. The stress components along the clamped boundary are compared with the solutions of finite element method (FEM) as shown in Figs 5, 7–9. (NISA User's Manual, 1997; Zienkiewicz and Taylor, 1991. The FEM model contained 1136 four-noded quadrilateral elements with the mesh as shown in Fig. 6. For conventional BEM, the problem needs many more elements to be discretized and the numerical solutions near to and on that boundary can not be given directly because of singular integration. It is shown in Fig. 7 that the difference of σ_x between present

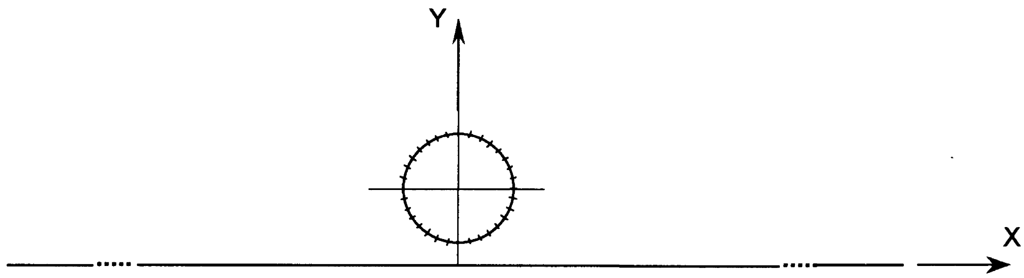

Fig. 5. Present CVBEM discretization.

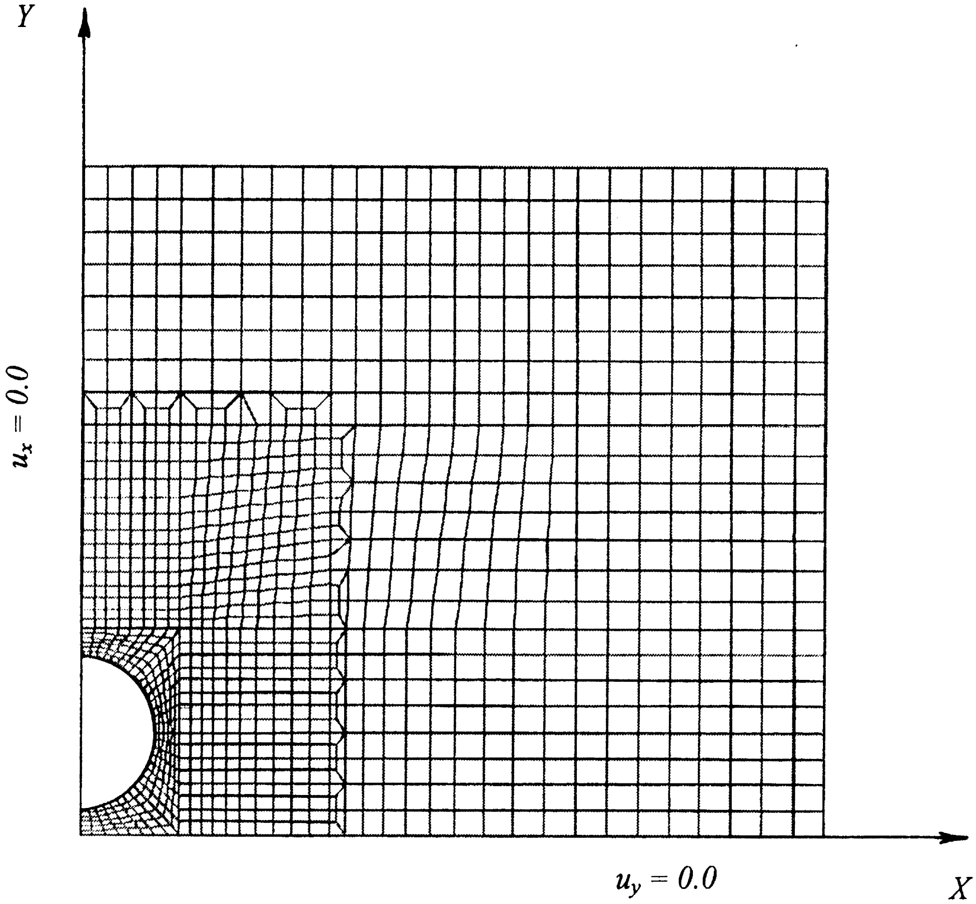

Fig. 6. Finite element discretization (1136 quad-4-noded elements).

method and FEM is about nine percent. The reason might be that in FEM, only element nodes along the clamped boundary are fixed which could not exactly express the total clamped boundary conditions. The comparisons at internal points among complex variable boundary element method (CVBEM) with 24 boundary elements, conventional BEM with 85 elements and FEM with 1136 finite elements are given in Figs 10–13. It can be found that the solutions obtained by CVBEM and BEM are in good agreement. Considering the symmetry of the problem, the shearing stress along symmetric axis should vanish. The present method gives shearing stress 5.1 E-6 MPa at point (0, 0) and 2.0 E-3 MPa at point (0, 7.3) which are more precise than that of conventional BEM [0.9 MPa at point (0, 7.3)] based on Kelvin solution and FEM [2.8 E-3 MPa at point (0, 0) and 0.4 MPa at point (0, 7.3)].

In the above example it is interesting to note that the ratio of the orders of coefficient matrices of the present method and conventional BEM is 1 : 3.5, and the ratio of computer interior capacities occupied 1 :12.5, which supports the high efficiency of the present method.

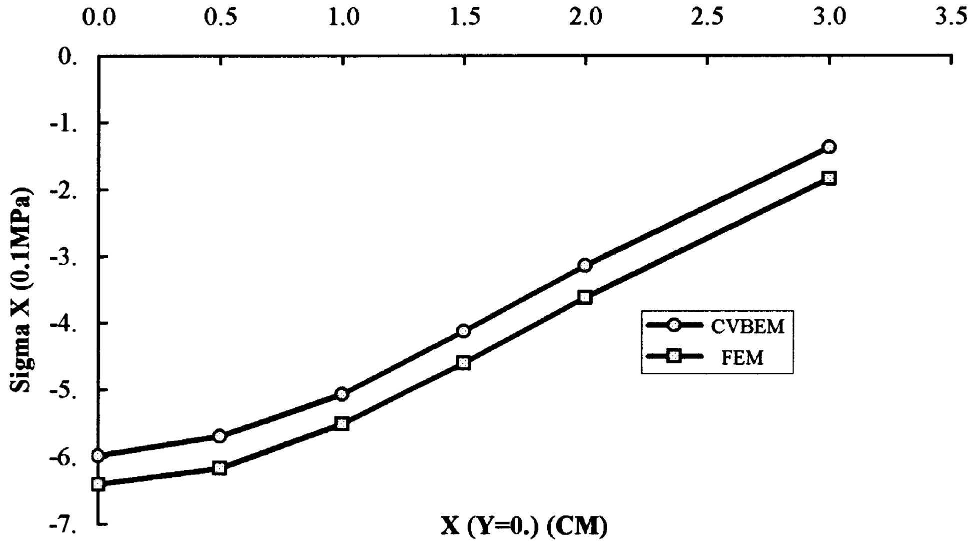

Fig. 7. Stress σ_{xx} at $(x, 0)$ on clamped boundary by CVBEM (24 elements) and FEM (1136 elements).

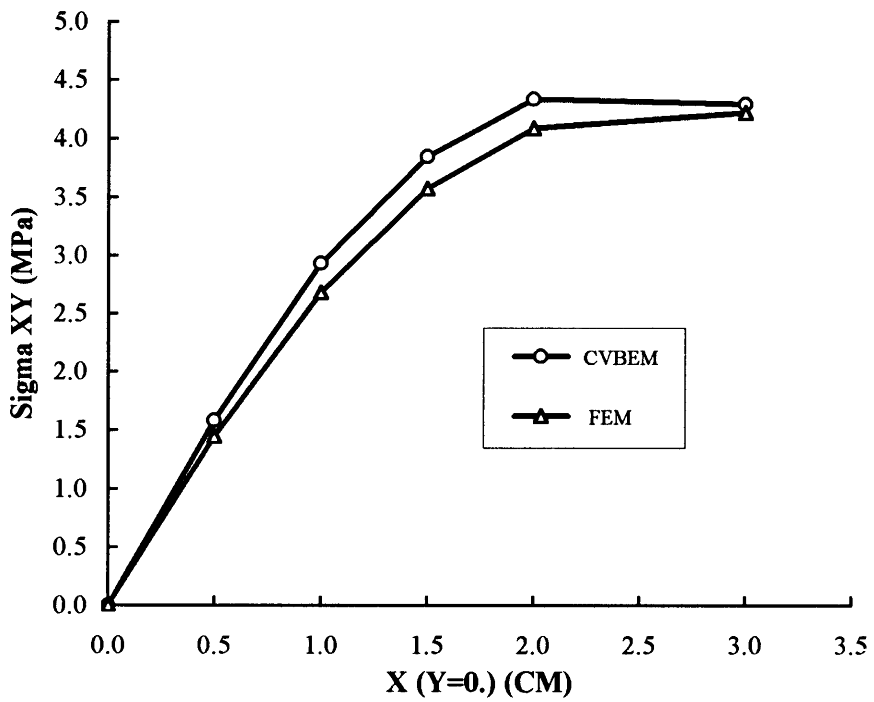

Fig. 8. Stress τ_{xy} at $(x, 0)$ on clamped boundary by CVBEM (24 elements) and FEM (1136 elements).

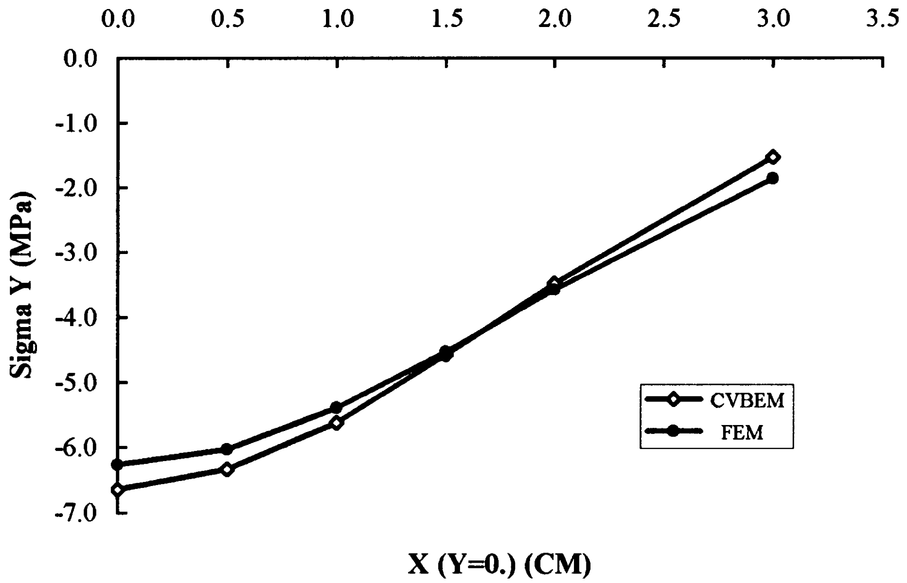

Fig. 9. Stress σ_{yy} at $(x, 0)$ on clamped boundary by CVBEM (24 elements) and FEM (1136 elements).

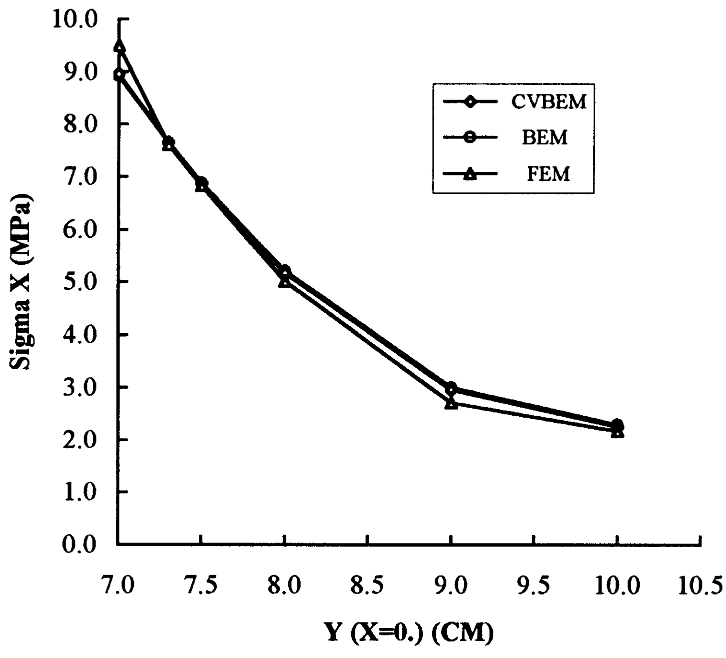

Fig. 10. Stress σ_{xx} at $(0, y)$ by CVBEM (24 elements), BEM (85 elements) and FEM (1136 elements).

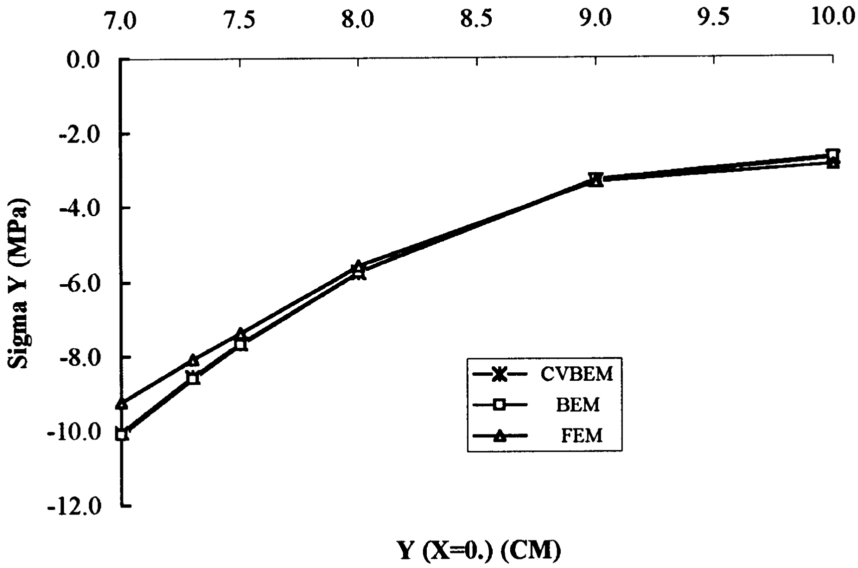

Fig. 11. Stress σ_{yy} at (0, y) by CVBEM (24 elements), BEM (85 elements) and FEM (1136 elements).

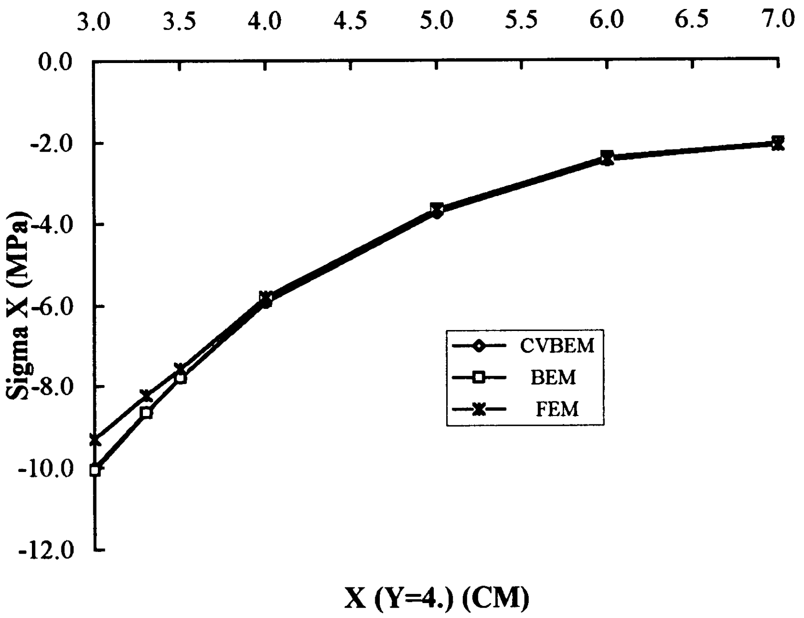

Fig. 12. Stress σ_{xx} at (x, 4.0) by CVBEM (24 elements), BEM (85 elements) and FEM (1136 elements).

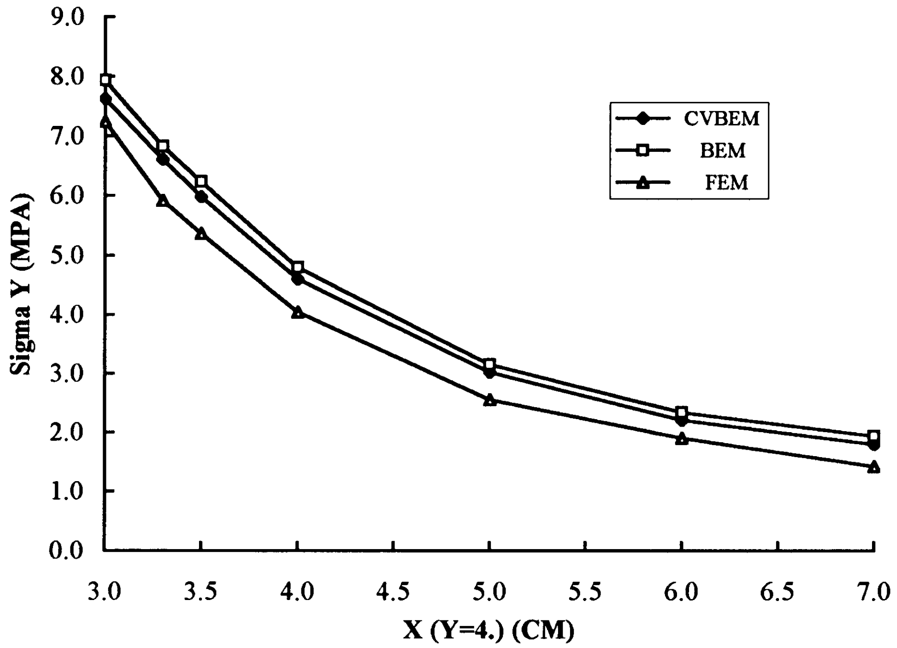

Fig. 13. Stress σ_{yy} at (x, 4.0) by CVBEM (24 elements), BEM (85 elements) and FEM (1136 elements).

7. Conclusions

A new complex variable fundamental solution for second kind of boundary value problems of half-plane has been derived and correspondent boundary integral equations have been established. The numerical procedure shows the advantages of present method over conventional BEM. The present BEM requires a relatively simple computer program, and improves the accuracy of numerical solutions for half-plane problems with clamped straight boundary. The random behavior near to and on the clamped boundary is eliminated. More efficient numerical solutions could be expected if linear or quadratic shape functions over boundary elements are employed with the appropriate but simple modifications in the computer program.

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